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ASYMMETRIC VIBRATIONS OF A CIRCULAR ELASTIC PLATE ON AN ELASTIC HALF SPACE†

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Abstract—The asymmetric problem of a vibrating circular elastic plate in frictionless contact with an elastic half space is solved by an integral equation method, where the contact stress appears as the unknown function. By a trigonometric expansion, the problem is reduced to a number of uncoupled two-dimensional problems. The radial variations of contact stresses and surface displacements are represented by polynomials, the coefficients of which are directly related by an infinite matrix that is a function of the vibration frequency. The results include a parametric study of the power input as a function of the vibration frequency of various plate stiffnesses and the normal component of the surface displacement field for simple excitation of the plate and passage of a plane Rayleigh wave.

1. INTRODUCTION

Problems concerning vibrating elastic plates in contact with an elastic medium are important in a number of practical applications. Perhaps the most interesting is that of a piezoelectric transducer mounted on a structure or test specimen. In that case special care is taken to obtain well defined contact conditions, and the frequencies are high enough to justify a detailed dynamic analysis. Such analyses have been performed for a number of contact problems for rigid discs, mainly by use of dual integral equations. A review has been given by Luco and Westmann[1]. The rigid disc formulation may be adequate in connection with soil-structure interaction problems, but in the case of piezoelectric transducers the stiffness moduli of the plate and the medium are of the same order of magnitude, and the plate must be treated as elastic. In this case it is more direct to formulate the problem in terms of one or more singular integral equations as demonstrated recently for the axisymmetric problem in [2]. Here the approach of [2] is generalized to asymmetric problems.

The problem under consideration is shown in Fig. 1. A circular elastic plate with radius a and thickness H is vibrating at the angular frequency ω while remaining in frictionless contact with an elastic half space. Shear modulus, Poisson's ratio and mass density are μ_p, ν_p, ρ_p and μ_h, ν_h, ρ_h for the plate and the half space, respectively. All physical quantities of the problem that vary with the time t contain this dependence through the factor $\exp(i\omega t)$. When $\partial/\partial t$ is replaced by the factor $i\omega$ and all dependent variables are allowed to take complex values, the time factor can be omitted. This convention is adopted here.

The main features of the method are the following. Because the interaction between the plate and the half space takes place through the normal contact stress $\sigma_{zz}(r, \theta)$ the problem is conveniently formulated in terms of this function. It is possible to decompose the original three-dimensional problem into a series of independent two-dimensional problems by use of a Fourier expansion in the angular variable θ. The contribution of order m is termed $\sigma_{zz}^m(r)$. When the expansion coefficients of the normal displacement of the contact surface are called $w^m(r)$, any linear theory of flexible plates that uncouples the different Fourier terms and assumes constant plate thickness leads to a relation of the form

$$w^m(r) = \int_0^a w_p^m(r, x)\, p^m(x)\, \mathrm{d}x, \quad m = 0, 1, .. \tag{1.1}$$

where $p^m(x)$ is the Fourier expansion coefficient of the total load $p(r, \theta) = q(r, \theta) + \sigma_{zz}(r, \theta)$ on the plate. This load includes any external normal excitation $q(r, \theta)$.

Similarly the normal displacements introduced in the half space by the contact stresses

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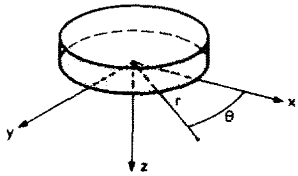

Fig. 1. Circular plate on elastic half space.

$\sigma_{zz}^m(r)$ are of the form

$$w^m(r) - \mathring{w}^m(r) = \int_0^a w_k^m(r, x)\, \sigma_{zz}^m(x)\, \mathrm{d}x, \quad m = 0, 1, .. \tag{1.2}$$

Here $\mathring{w}^m(r)$ are the expansion coefficients of any normal surface displacement that may be present in the absence of the plate. This is the most interesting kind of excitation in connection with piezoelectric transducers.

In the relations (1.1) and (1.2) the functions $q^m(r)$ and $\mathring{w}^m(r)$, $m = 0, 1, \ldots$ are prescribed, while $w^m(r)$ and $\sigma_{zz}^m(r)$, $m = 0, 1..$ are unknown. The condition of frictionless contact over the full plate contact surface allows elimination of $w^m(r)$ leading to the series of integral equations

$$\int_0^a [w_k^m(r, x) - w_p^m(r, x)]\sigma_{zz}^m(x)\, \mathrm{d}x = -\mathring{w}^m(r) + \int_0^a w_p^m(r, x)\, q^m(x)\, \mathrm{d}x,$$

$$m = 0, 1, .. \tag{1.3}$$

These integral equations are of the first kind, but a logarithmic singularity in $w_k^m(r, x)$ leads to square root singularities in $\sigma_{zz}^m(r)$ at $r = a$ and a numerically stable solution. On the other hand, the singularities call for special numerical integration formulae. After derivation of the necessary influence functions $w_k^m(r, x)$ and $w_p^m(r, x)$ in Sections 2 and 3 the numerical solution technique is treated in Section 4. Examples dealing with the rocking mode of the plate, and the normal deformation of the half space due to plate excitation and wave passage are presented in Section 5. Results for the limiting case of a rigid disc are also included.

2. THE HALF SPACE

A cylindrical coordinate system is introduced with the elastic half space occupying $z \geqslant 0$, Fig. 1. The cylindrical displacement components are u, v and w. As is well known the equations of dynamic elasticity are satisfied by displacements derived from the scalar potential Φ and the vector potential $\boldsymbol{\Psi}$ by the formulae [3]

$$u = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi_z}{\partial \theta} - \frac{\partial \Psi_\theta}{\partial z} \tag{2.1}$$

$$v = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\partial \Psi_r}{\partial z} - \frac{\partial \Psi_z}{\partial r} \tag{2.2}$$

$$w = \frac{\partial \Phi}{\partial z} + \frac{1}{r}\frac{\partial (r\Psi_\theta)}{\partial r} - \frac{1}{r}\frac{\partial \Psi_r}{\partial \theta} \tag{2.3}$$

where the potentials satisfy

$$\nabla^2 \Phi = \frac{1}{c_L^2}\frac{\partial^2 \Phi}{\partial t^2} \tag{2.4}$$

$$\nabla^2 \Psi = \frac{1}{c_T^2} \frac{\partial^2 \Phi}{\partial t^2} \tag{2.5}$$

c_L and c_T are the velocities of longitudinal and transverse waves, respectively[3].

In (2.5) the equations for Ψ_r and Ψ_θ are coupled. It is therefore convenient to express Ψ_r and Ψ_θ in terms of a scalar potential Λ in the form

$$\Psi_r = \frac{1}{r} \frac{\partial \Lambda}{\partial \theta} \tag{2.6}$$

$$\Psi_\theta = -\frac{\partial \Lambda}{\partial r} \tag{2.7}$$

where Λ satisfies the equation

$$\nabla^2 \Lambda = \frac{1}{c_T^2} \frac{\partial^2 \Lambda}{\partial t^2}. \tag{2.8}$$

A term of gradient form has been omitted from (2.6) and (2.7), as it can be included in Ψ_z. In the following the subscript z will be dropped, and the displacement field will be derived from the three scalar potentials Φ, Λ and Ψ satisfying (2.4), (2.8) and

$$\nabla^2 \Psi = \frac{1}{c_T^2} \frac{\partial^2 \Psi}{\partial t^2}. \tag{2.9}$$

The following is restricted to vibrations with angular frequency ω.

Displacements, stresses and potential functions will be written in complex notation and will contain the factor $\exp(i\omega t)$, without this factor appearing explicitly in the formulae. The potential functions are expanded as†

$$\Phi(\eta, \theta, \zeta) = \sum_{m=0}^{\infty} \Phi^m(\eta, \zeta) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \tag{2.10}$$

$$\Lambda(\eta, \theta, \zeta) = \sum_{m=0}^{\infty} \Lambda^m(\eta, \zeta) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \tag{2.11}$$

$$\Psi(\eta, \theta, \zeta) = \sum_{m=0}^{\infty} \Psi^m(\eta, \zeta) \begin{Bmatrix} \sin(m\theta) \\ -\cos(m\theta) \end{Bmatrix} \tag{2.12}$$

where $\eta = r/a$ and $\zeta = z/a$ are dimensionless coordinates. a is a characteristic length that will be taken equal to the radius of the plate. When these expansions are substituted into (2.4), (2.8) and (2.9), use of the Hankel transform yields the integral representations

$$\Phi^m(\eta, \zeta) = a^2 \int_0^\infty A^m(s) s \, e^{-\zeta\alpha(s)} J_m(\eta s) \, ds \tag{2.13}$$

$$\Lambda^m(\eta, \zeta) = a^3 \int_0^\infty B^m(s) \, e^{-\zeta\beta(s)} J_m(\eta s) \, ds \tag{2.14}$$

$$\Psi^m(\eta, \zeta) = a^2 \int_0^\infty C^m(s) s \, e^{-\zeta\beta(s)} J_m(\eta s) \, ds \tag{2.15}$$

where $J_m(\;)$ is the Bessel function of the first kind of order m. $A^m(s)$, $B^m(s)$ and $C^m(s)$ are

†Use of the complex form of the Fourier expansion would require introduction of an imaginary unit different from i already used in the time factor.

arbitrary complex functions, while

$$\alpha(s)=\begin{cases}(s^2-h^2)^{1/2} & s>h\\ i(h^2-s^2)^{1/2} & |s|\leq h\\ -(s^2-h^2)^{1/2} & s<-h\end{cases} \tag{2.16}$$

and

$$\beta(s)=\begin{cases}(s^2-k^2)^{1/2} & s>k\\ i(k^2-s^2)^{1/2} & |s|\leq k\\ -(s^2-k^2)^{1/2} & s<-k\end{cases} \tag{2.17}$$

The dimensionless parameters h and k are the relative wavenumbers of longitudinal and transverse waves, respectively.

$$h^2=\left(\frac{\omega a}{c_L}\right)^2=\frac{1}{2}\frac{1-2\nu_h}{1-\nu_h}\frac{\omega^2a^2\rho_h}{\mu_h} \tag{2.18}$$

$$k^2=\left(\frac{\omega a}{c_T}\right)^2=\frac{\omega^2a^2\rho_h}{\mu_h} \tag{2.19}$$

ν_h, μ_h, ρ_h are Poisson's ratio, the shear modulus and the mass density of the half space. The three independent solutions given by (2.13), (2.14) and (2.15) correspond to those obtained by direct integration by Sezawa [4].

The displacement components $w(\eta, \theta, \zeta)$ and $u(\eta, \theta, \zeta)$ are expanded like (2.10) in terms of $w^m(\eta, \zeta)$ and $u^m(\eta, \zeta)$, while $v(\eta, \theta, \zeta)$ is expanded like (2.12) in terms of $v^m(\eta, \zeta)$. The expansion coefficients follow from (2.1)–(2.3).

$$w^m(\eta,\zeta)=\frac{\partial}{\partial z}\Phi^m(\eta,\zeta)+\left(\frac{\partial^2}{\partial z^2}+\frac{k^2}{a^2}\right)\Lambda^m(\eta,\zeta) \tag{2.20}$$

$$u^m(\eta,\zeta)\pm v^m(\eta,\zeta)=\left(\frac{\partial}{\partial\eta}\mp\frac{m}{\eta}\right)\left[\Phi^m(\eta,\zeta)+\frac{\partial}{\partial z}\Lambda^m(\eta,\zeta)\mp\Psi^m(\eta,\zeta)\right]. \tag{2.21}$$

By use of the relation

$$\left(\frac{\partial}{\partial\eta}\mp\frac{m}{\eta}\right)J_m(\eta s)=\mp\, s\,J_{m\pm1}(\eta s) \tag{2.22}$$

integral representations for the displacement expansion coefficients are obtained in the form

$$w^m(\eta,\zeta)/a=\int_0^\infty\left\{-A^m(s)s\,\alpha(s)\,e^{-\zeta\alpha(s)}+B^m(s)s^2\,e^{-\zeta\beta(s)}\right\}J_m(\eta s)\,ds \tag{2.23}$$

$$[u^m(\eta,\zeta)\pm v^m(\eta,\zeta)]/a=\int_0^\infty\{\mp A^m(s)s^2\,e^{-\zeta\alpha(s)}\pm B^m(s)s\,\beta(s)\,e^{-\zeta\beta(s)}$$

$$+\,C^m(s)s^2\,e^{-\zeta\beta(s)}\}\,J_{m\pm1}(\eta s)\,ds. \tag{2.24}$$

When the stress components $\sigma_{zz}(\eta, \theta, \zeta)$, $\sigma_{rz}(\eta, \theta, \zeta)$ and $\sigma_{\theta z}(\eta, \theta, \zeta)$ are expanded like (2.10), (2.11) and (2.12), respectively, integral representations for the expansion coefficients follow from differentiation of (2.23) and (2.24) by Hooke's law.

$$\sigma_{zz}^m(\eta,\zeta)/\mu_h=\int_0^\infty\{A^m(s)\,(2s^2-k^2)s\,e^{-\zeta\alpha(s)}-B^m(s)2s^2\beta(s)\,^{-\zeta\beta(s)}\}\,J_m(\eta s)\,ds \tag{2.25}$$

$$[\sigma_{rz}^m(\eta,\zeta) \pm \sigma_{\theta z}^m(\eta,\zeta)]/\mu_h = \int_0^\infty \{\pm A^m(s) 2s^2\alpha(s)\,e^{-\zeta\alpha(s)}$$

$$\mp B^m(s)(2s^2-k^2)s\,e^{-\zeta\beta(s)} - C^m(s)s^2\beta(s)^{-\zeta\beta(s)}\} J_{m\pm 1}(\eta s)\,ds. \quad (2.26)$$

It is noted that the index m only appears in the integral representations (2.23)–(2.26) through the order of the Bessel functions. For $m = 0$ the axisymmetric problem described by $A^0(s)$ and $B^0(s)$ uncouples from the torsion problem described by $C^0(s)$.

If either the normal stress or the shear stress vector on the surface $z = 0$ vanishes, contact problems for the circle can be formulated by eliminating two or one of the series of arbitrary functions $A^m(s)$, $B^m(s)$ and $C^m(s)$ and then proceeding to an infinite series of dual or singular integral equations. In the present context, where compability of the representations of surface stresses and displacements with those of the plate is essential, the use of a singular integral relating surface stresses to surface displacements is the natural choice.

In the frictionless contact problem use of eqn (2.26) with $\zeta = 0$ yields

$$A^m(s) = \frac{2s^2-k^2}{2s\alpha(s)} B^m(s), \qquad m = 0, 1, .. \quad (2.27)$$

$$C^m(s) = 0, \qquad m = 0, 1, .. \quad (2.28)$$

It is seen that the normal load does not excite oscillations associated with the potential $\Psi(\eta, \theta, \zeta)$. Substitution of these relations into (2.25) and inversion give the functions $B^m(s)$ in terms of the expansion coefficients of the contact stress coefficients $\sigma_{zz}^m(\eta, 0)$.

$$B^m(s) = \frac{s\alpha(s)}{(s^2-1/2k^2)^2 - s^2\alpha(s)\beta(s)} \frac{1}{2\mu_h} \int_0^\infty \sigma_{zz}^m(\eta, 0)\eta J_m(\eta s)\,d\eta. \quad (2.29)$$

In the absence of normal stress for $r > a$ the upper limit of the integral can be changed to 1. By use of (2.27)–(2.29) in the integral representations (2.23)–(2.26) the displacements and stresses in the half space are given in terms of the yet unknown contact stresses. In particular the normal surface displacement due to the contact stress $\sigma_{zz}^m(\xi, 0)$ is

$$w^m(\eta, 0) = \frac{ak^2}{4\mu_h} \int_0^\infty \frac{\alpha(s)s J_m(\eta s)}{(s^2-1/2k^2)^2 - s^2\alpha(s)\beta(s)} \int_0^1 \sigma_{zz}^m(\xi, 0)\xi J_m(\xi s)\,d\xi\,ds. \quad (2.30)$$

This is the integral $\int_0^1 w_h^m(\eta, \xi)\,\sigma_{zz}^m(\xi, 0)\,d\xi$ from (1.2). Its numerical evaluation is dealt with after treating the corresnonding integral for the elastic plate.

3. THE ELASTIC PLATE

The elastic plate is described by the equations given by Mindlin[5] including the effects of transverse shear deformations and rotatory inertia. Both effects are accounted for by including the angular displacements (ψ_r, ψ_θ) as independent variables. The equations of motion are conveniently formulated in terms of the transverse displacement of the mid-plane $w(r, \theta)$ and two potentials $\phi(r, \theta)$ and $\chi(r, \theta)$ that determine ψ_r and ψ_θ by the relations

$$\psi_r = \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\chi}{\partial\theta} \quad (3.1)$$

$$\psi_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{\partial\chi}{\partial r}. \quad (3.2)$$

The equations of motion follow from the eqns (16) of Mindlin's paper[5].

$$D\nabla_2^4\phi - \kappa^2\mu_p H\nabla_2^2(w+\phi) = \frac{\rho_p H^3}{12}\frac{\partial^2}{\partial t^2}\nabla_2^2\phi \quad (3.3)$$

$$\frac{1-\nu_p}{2} D\,\nabla_2^4\chi - \kappa^2\mu_p H\nabla_2^2\chi = \frac{\rho_p H^3}{12}\frac{\partial^2}{\partial t^2}\nabla_2^2\chi \tag{3.4}$$

$$\nabla_2^2(w+\phi) = \frac{\rho_p}{\kappa^2\mu_p}\frac{\partial^2}{\partial t^2}w - \frac{p}{\kappa^2\mu_p H} \tag{3.5}$$

H is the plate thickness, and ν_p, μ_p and ρ_p are Poisson's ratio, the shear modulus and the mass density of the plate. $D = H^3\mu_p/6(1-\nu_p)$ is the flexural rigidity and κ^2 the shear stiffness coefficient accounting for nonuniformity of the transverse shear stress distribution. A parabolic distribution leads to $\kappa^2 = 5/6$, while matching of the thickness shear frequency gives $\kappa^2 = \pi^2/12$. Finally p is the normal load per unit area.

It is convenient to introduce the dimensionless parameters

$$\tau = H/a \tag{3.6}$$

$$\omega_S^2 = \frac{12\kappa^2\mu_p}{H^2\rho_p} \tag{3.7}$$

$$\omega_E^2 = \frac{2\mu_p}{H^2(1-\nu_p)\rho_p} \tag{3.8}$$

the dimensionless variables $\eta = r/a$, $\xi = x/a$ and the differential operator

$$\nabla_\eta^2 = a^2\nabla_2^2 = \frac{1}{\eta}\frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2} \tag{3.9}$$

a is the radius of the plate.

After introducing the time dependence through the angular frequency ω the differential equations take the form

$$\tau^2\nabla_\eta^2\phi - \left(\frac{\omega_S}{\omega_E}\right)^2(w+\phi) + \left(\frac{\omega_S}{\omega_E}\right)^2\phi = 0 \tag{3.10}$$

$$\tau^2\nabla_\eta^2(w+\phi) + 12\left(\frac{\omega}{\omega_S}\right)^2 w + \left(\frac{\omega_E}{\omega_S}\right)^2\frac{H^4 p}{D} = 0 \tag{3.11}$$

$$\tau^2\nabla_\eta^2\chi - 12\kappa^2\left(1-\left(\frac{\omega}{\omega_S}\right)^2\right)\chi = 0. \tag{3.12}$$

It is noted that the last equation uncouples, and that modes associated with the potential χ are not excited *directly* by a normal load.

The potential ϕ is most conveniently expressed in terms of w by (3.11), and elimination yields

$$\left[\left(\tau^2\nabla_\eta^2 + 12\left(\frac{\omega}{\omega_S}\right)^2\right)\left(\tau^2\nabla_\eta^2 + \left(\frac{\omega}{\omega_E}\right)^2\right) - 12\left(\frac{\omega}{\omega_E}\right)^2\right]w$$

$$= \left[1 - \tau^2\left(\frac{\omega_E}{\omega_S}\right)^2\nabla_\eta^2 - \left(\frac{\omega}{\omega_S}\right)^2\right]\frac{H^4 p}{D}. \tag{3.13}$$

The eqns (3.13) and (3.12) are of the form

$$[\nabla_\eta^2 + \delta_1^2(\omega)]\,[\nabla_\eta^2 + \delta_2^2(\omega)]w = \left[1 - \tau^2\left(\frac{\omega_E}{\omega_S}\right)^2\nabla_\eta^2 - \left(\frac{\omega}{\omega_S}\right)^2\right]\frac{a^4 p}{D} \tag{3.14}$$

$$[\nabla_\eta^2 + \delta_3^2(\omega)]\chi = 0 \tag{3.15}$$

where

$$2\tau^2\delta_{1,2}^2(\omega) = 12\left(\frac{\omega}{\omega_S}\right)^2 + \left(\frac{\omega}{\omega_E}\right)^2 \pm \sqrt{\left(\left[12\left(\frac{\omega}{\omega_S}\right)^2 - \left(\frac{\omega}{\omega_E}\right)^2\right]^2 + 48\left(\frac{\omega}{\omega_E}\right)^2\right)} \tag{3.16}$$

$$\tau^2\delta_3^2(\omega) = 12\kappa^2\left[\left(\frac{\omega}{\omega_S}\right)^2 - 1\right] \tag{3.17}$$

$\delta_1^2(\omega)$ is positive for all values of ω, whereas $\delta_2^2(\omega)$ and $\delta_3^2(\omega)$ are negative for $0 < \omega < \omega_S$ and positive for $\omega_S < \omega$. For $0 < \omega < \omega_S$ the notation $\delta_j^*(\omega) = |i\delta_j(\omega)|$, $j = 2, 3$ is introduced.

The moments and shear forces are given in terms of the displacement functions w, ψ_r and ψ_θ as

$$M_r = D\left[\frac{\partial\psi_r}{\partial r} + \frac{\nu_p}{r}\left(\psi_r + \frac{\partial\psi_\theta}{\partial\theta}\right)\right] \tag{3.18}$$

$$M_\theta = D\left[\frac{1}{r}\left(\psi_r + \frac{\partial\psi_\theta}{\partial\theta}\right) + \nu_p\frac{\partial\psi_r}{\partial r}\right] \tag{3.19}$$

$$M_{r\theta} = \frac{1-\nu_p}{2}D\left[\frac{1}{r}\left(\frac{\partial\psi_r}{\partial\theta} - \psi_\theta\right) + \frac{\partial\psi_\theta}{\partial r}\right] \tag{3.20}$$

$$Q_r = \kappa^2\mu_p H\left[\psi_r + \frac{\partial w}{\partial r}\right] \tag{3.21}$$

$$Q_\theta = \kappa^2\mu_p H\left[\psi_\theta + \frac{1}{r}\frac{\partial w}{\partial\theta}\right]. \tag{3.22}$$

Now the solutions corresponding to concentrated ring loads of magnitude $\{\cos(m\theta)/\sin(m\theta)\}$, $m = 0, 1, \ldots$ at $x = \xi a$ on a free plate are derived. These solutions are of the form

$$w_p(\eta, \xi, \theta) = w_p^m(\eta, \xi)\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{3.23}$$

$$\phi_p(\eta, \xi, \theta) = \phi_p^m(\eta, \xi)\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{3.24}$$

$$\chi_p(\eta, \xi, \theta) = \chi_p^m(\eta, \xi)\begin{Bmatrix}\sin(m\theta)\\ -\cos(m\theta)\end{Bmatrix} \tag{3.25}$$

and similar for the moments and shear forces. The solution of order m satisfies the homogeneous equations (3.10)–(3.12) for $0 \le \eta < \xi$ and $\xi < \eta \le 1$ and the discontinuity condition

$$Q_r^m(\xi-, \xi) - Q_r^m(\xi+, \xi) = 1. \tag{3.26}$$

Furthermore, $w_p(\eta, \xi)$, $\psi_r(\eta, \xi)$, $\psi_\theta(\eta, \xi)$, $M_r(\eta, \xi)$ and $M_{r\theta}(\eta, \xi)$ are continuous at $\eta = \xi$ and $M_r(1, \xi) = M_{r\theta}(1, \xi) = Q_r(1, \xi) = 0$.

It is convenient to replace the continuity conditions on ψ_r, ψ_θ, M_r and $M_{r\theta}$ with continuity conditions on the potentials ψ and χ. It follows from (3.18) and (3.20) that the continuity conditions on M_r and $M_{r\theta}$ can be replaced by continuity conditions on $\partial\psi_r/\partial\eta$ and $\partial\psi_\theta/\partial\eta$. By differentiation of (3.1) and (3.2) $\nabla_\eta^2\phi_p$ and $\nabla_\eta^2\chi_p$ are expressed in terms of continuous functions, and thus ϕ_p and χ_p are continuous by (3.10) and (3.11). Finally (3.1) and (3.2) then imply continuity of $\partial\phi_p/\partial\eta$ and $\partial\chi_p/\partial\eta$. Thus the continuity conditions may be imposed on w_p, ϕ_p, $\partial\phi_p/\partial\eta$, χ_p and $\partial\chi_p/\partial\eta$, while the discontinuity condition (3.26) can be written as

$$\frac{\partial}{\partial\eta}w_p^m(\xi-, \xi) - \frac{\partial}{\partial\eta}w_p^m(\xi+, \xi) = \frac{1}{\tau\kappa^2\mu_p}. \tag{3.27}$$

For $0 < \omega < \omega_S$ the solution is

$$w_p^m(\eta, \xi) = C_1^m(\xi) J_m(\delta_1 \eta) + C_2^m(\xi) I_m(\delta_2^* \eta), \quad 0 \le \eta \le \xi \tag{3.28}$$

$$+ C_5^m(\xi) Y_m(\delta_1 \eta) + C_6^m(\xi) K_m(\delta_2^* \eta), \quad \xi \le \eta \le 1 \tag{3.29}$$

and

$$\chi_p^m(\eta, \xi) = C_7^m(\xi) I_m(\delta_3^* \eta), \qquad 0 \le \eta \le 1. \tag{3.30}$$

The potentials $\phi^m(\xi, \eta)$ follow from (3.10) in the form

$$[\phi_p^m(\eta, \xi) + w_p^m(\eta, \xi)] = \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2 \left[\frac{C_1^m(\xi)}{\delta_1^2} J_m(\delta_1 \eta) + \frac{C_2^m(\xi)}{\delta_2^2} I_m(\delta_2^* \eta)\right], \qquad 0 \le \eta \le \xi \tag{3.31}$$

$$[\phi_p^m(\eta, \xi) + w_p^m(\eta, \xi)] = \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2 \left[\frac{C_3^m(\xi)}{\delta_1^2} J_m(\delta_1 \eta) + \frac{C_4^m(\xi)}{\delta_2^2} I_m(\delta_2^* \eta)\right.$$

$$\left. + \frac{C_5(\xi)}{\delta_1^2} Y_m(\delta_1 \eta) + \frac{C_6^m(\xi)}{\delta_2^2} K_m(\delta_2^* \eta)\right], \qquad \xi \le \eta \le 1. \tag{3.32}$$

The arbitrary functions $C_j^m(\xi)$, $j = 1, \ldots, 7$ are determined from the following system of equations. The first two equations express continuity of w^m and $(w^m + \phi^m)$, and the next two prescribe the discontinuity (3.27) of their derivatives at $\eta = \xi$. The last three are the homogeneous boundary conditions at $\eta = 1$.

For $\omega_S < \omega$ the eqns (3.33) must be modified as follows. The coefficients of $C_2^m(\xi)$, $C_4^m(\xi)$ and $C_6^m(\xi)$ are identical to those of $C_1^m(\xi)$, $C_3^m(\xi)$ and $C_5^m(\xi)$, respectively, but with δ_1 replaced by δ_2. In the last column of the matrix $I_m(\ )$ must be replaced by $J_m(\ )$, and δ_3^* must be replaced by δ_3.

For any given value of ξ solution of the equations (3.33) and substitution of $C_j^m(\xi)$ in (3.28) and (3.29) yield the kernel $w_p^m(\eta, \xi)$ of the integral in (1.1). The necessary modifications of the procedure for rigid plates are discussed in connection with the numerical solution in the next section.

4. NUMERICAL SOLUTION TECHNIQUE

The numerical solution follows a technique developed in[6] and[2]. It rests on the fact that the kernel of the integral equation (1.3) is bounded apart from a contribution of the form

$$-\frac{a}{\mu_h}\int_0^\infty J_m(\eta s)\int_0^1 \sigma_{zz}^m(\xi, 0)\xi J_m(\xi s)\,d\xi\,ds \tag{4.1}$$

identified by introducing the following limit in (2.30).

$$\lim_{s/k \to \infty} \frac{k^2}{4(1-\nu_h)} \frac{s\alpha(s)}{(s^2 - 1/2k^2)^2 - s^2\alpha(s)\beta(s)} = -1. \tag{4.2}$$

When the contact stress $\sigma_{zz}^m(\xi, 0)$ is expanded as a series, the integral (4.1) defines a series expansion suitable for the deformations. As shown in [6] a convenient pair of expansions in terms of orthogonal polynomials is

$$-\frac{1-\nu_h}{\mu_h}\sigma_{zz}^m(\xi, 0) = (1-\xi^2)^{-1/2}\sum_{j=0}^\infty S_j^m\, P_{m+2j}^m(\sqrt{(1-\xi^2)})P_{m+2j}^m(0) \tag{4.3}$$

$$w^m(\eta, 0) = a\sum_{l=0}^\infty W_l^m \frac{(2l)!}{(2m+2l)!} P_{m+2l}^m(0)\, P_{m+2l}^m(\sqrt{(1-\eta^2)}) \tag{4.4}$$

$$
\begin{bmatrix}
J_m(\delta_1\xi) & I_m(\delta_2^*\xi) & -J_m(\delta_1\xi) & -I_m(\delta_2^*\xi) & -Y_m(\delta_1\xi) & -K_m(\delta_2^*\xi) & 0 \\
\frac{1}{\delta_1^2} J_m(\delta_1\xi) & \frac{1}{\delta_2^2} I_m(\delta_2^*\xi) & -\frac{1}{\delta_1^2} J_m(\delta_1\xi) & -\frac{1}{\delta_2^2} I_m(\delta_2^*\xi) & -\frac{1}{\delta_1^2} Y_m(\delta_1\xi) & -\frac{1}{\delta_2^2} K_m(\delta_2^*\xi) & 0 \\
\delta_1 J'_m(\delta_1\xi) & \delta_2^* I'_m(\delta_2^*\xi) & -\delta_1 J'_m(\delta_1\xi) & -\delta_2^* I'_m(\delta_2^*\xi) & -\delta_1 Y'_m(\delta_1\xi) & -\delta_2^* K'_m(\delta_2^*\xi) & 0 \\
\frac{1}{\delta_1} J'_m(\delta_1\xi) & -\frac{1}{\delta_2^*} I'_m(\delta_2^*\xi) & -\frac{1}{\delta_1} J'_m(\delta_1\xi) & \frac{1}{\delta_2^*} I'_m(\delta_2^*\xi) & -\frac{1}{\delta_1} Y'_m(\delta_1\xi) & \frac{1}{\delta_2^*} K'_m(\delta_2^*\xi) & 0 \\
0 & 0 & -\frac{1}{\delta_1} J'_m(\delta_1) & \frac{1}{\delta_2^*} I'_m(\delta_2^*) & -\frac{1}{\delta_1} Y'_m(\delta_1) & \frac{1}{\delta_2^*} K'_m(\delta_2^*) & -\frac{m}{12}\left(\frac{\tau\omega}{\omega_S}\right)^2 I_m(\delta_3^*) \\
0 & 0 &
\begin{array}{l}\left[\delta_1^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[\left(1-m^2\frac{1-\nu_p}{\delta_1^2}\right) J_m(\delta_1) \right.\\ \left. + \frac{1-\nu_p}{\delta_1} J'_m(\delta_1)\right]\end{array} &
\begin{array}{l}\left[\delta_2^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[\left(1-m^2\frac{1-\nu_p}{\delta_2^2}\right) I_m(\delta_2^*) \right.\\ \left. - \frac{1-\nu_p}{\delta_2^*} I'_m(\delta_2^*)\right]\end{array} &
\begin{array}{l}\left[\delta_1^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[\left(1-m^2\frac{1-\nu_p}{\delta_1^2}\right) Y_m(\delta_1) \right.\\ \left. + \frac{1-\nu_p}{\delta_1} Y'_m(\delta_1)\right]\end{array} &
\begin{array}{l}\left[\delta_2^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[\left(1-m^2\frac{1-\nu_p}{\delta_2^2}\right) K_m(\delta_2^*) \right.\\ \left. - \frac{1-\nu_p}{\delta_2^*} K'_m(\delta_2^*)\right]\end{array} &
\begin{array}{l} -m(1-\nu_p)\times \\ \left[I_m(\delta_3^*) - \delta_3^* I'_m(\delta_3^*)\right]\end{array} \\
0 & 0 &
\begin{array}{l}\frac{m}{\delta_1^2}\left[\delta_1^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[J_m(\delta_1) - \delta_1 J'_m(\delta_1)\right]\end{array} &
\begin{array}{l}\frac{m}{\delta_2^2}\left[\delta_2^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[I_m(\delta_2^*) - \delta_2^* I'_m(\delta_2^*)\right]\end{array} &
\begin{array}{l}\frac{m}{\delta_1^2}\left[\delta_1^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[Y_m(\delta_1) - \delta_1 Y'_m(\delta_1)\right]\end{array} &
\begin{array}{l}\frac{m}{\delta_2^2}\left[\delta_2^2 - \frac{12}{\tau^2}\left(\frac{\omega}{\omega_S}\right)^2\right]\times \\ \left[K_m(\delta_2^*) - \delta_2^* K'(\delta_2^*)\right]\end{array} &
\begin{array}{l} -\left[\left(\tfrac{1}{2}\delta_3^2 - m^2\right) I_m(\delta_3^*)\right. \\ \left. + \delta_3^* I'_m(\delta_3^*)\right]\end{array}
\end{bmatrix}
\begin{bmatrix} C_1^m(\xi) \\ C_2^m(\xi) \\ C_3^m(\xi) \\ C_4^m(\xi) \\ C_5^m(\xi) \\ C_6^m(\xi) \\ C_7^m(\xi) \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \frac{1}{\kappa^2\tau\mu_p} \\ \frac{1}{12}\frac{\tau}{\kappa^2\mu_p}\left(\frac{\omega_S}{\omega}\right)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad (3.33)
$$

P_n^m is the Legendre function of the first kind[7], and the normalization factor in (4.3) is

$$P_{m+2j}^m(0)=\frac{(-1)^{m+j}}{2^{m+2j}}\frac{(2m+2j)!}{j!(m+j)!} \tag{4.5}$$

$P_{m+2j}^m(\sqrt{(1-\xi^2)})$ is ξ^m times a polynomial of degree j in ξ^2. It is noted that the contact stress $\sigma_{zz}^m(\xi,0)$ exhibits a square root singularity represented by the factor $(1-\xi^2)^{-1/2}$. Expansion of the forms (4.3) and (4.4) were used for a static half space problem by Gladwell[8].

In the calculations the following result from[6] is needed.

$$\int_0^1 \frac{P_{m+2j}^m(\sqrt{(1-\xi^2)})}{P_{m+2j}^m(0)}(1-\xi^2)^{-1/2}\,\xi J_m(\xi s)\,d\xi=(-1)^j j_{m+2j}(s) \tag{4.6}$$

$j_m(\ )$ is the spherical Bessel function of the first kind of order m[9]. With (4.3) and this result (2.30) takes the form

$$w^m(\eta,0)=-\sum_{j=0}^{\infty}(-1)^j S_j^m\frac{k^2a}{4(1-\nu_h)}\int_0^{\infty}\frac{\alpha(s)sJ_m(\eta s)j_{m+2j}(s)}{(s^2-1/2k^2)^2-s^2\alpha(s)\beta(s)}ds. \tag{4.7}$$

The denominator has one root on the positive real axis corresponding to Rayleigh waves. When the path of integration is indented into the upper half plane at the pole the Rayleigh waves diverge thus satisfying the radiation condition at infinity.

By substitition of the expansion (4.4) and use of the orthogonality relation for the Legendre functions[7]

$$\int_0^1 P_{m+2l}^m(\sqrt{(1-\eta^2)})P_{m+2j}^m(\sqrt{(1-\eta^2)})\frac{\eta\,d\eta}{\sqrt{(1-\eta^2)}}=\begin{cases}0 & j\neq l\\ \dfrac{1}{2m+4l+1}\dfrac{(2m+2l)!}{(2l)!} & j=l\end{cases} \tag{4.8}$$

the following formula for W_l^m in terms of the coefficients S_j^m, $j=0,1,\ldots$ is obtained for the surface deformation of the half space due to contact stresses.

$$W_l^m=(2m+4l+1)\sum_{j=0}^{\infty}(-1)^{j+l}S_j^m A_{jl}^m \tag{4.9}$$

where A_{jl}^m is the symmetric matrix

$$A_{jl}^m=-\frac{k^2}{4(1-\nu_h)}\int_0^{\infty}\frac{\alpha(s)s}{(s^2-1/2k^2)^2-s^2\alpha(s)\beta(s)}j_{m+2j}(s)j_{m+2l}(s)\,ds. \tag{4.10}$$

Direct numerical evaluation of (4.10) would be rather inconvenient. By contour integration in the complex plane the following expression is found (see the Appendix of [2]).

$$\begin{aligned}A_{jl}^m=&-\frac{i\pi k}{4(1-\nu_h)}\frac{\xi_R^2(\xi_R^2-\gamma^2)\sqrt{(\xi_R^2-1)}}{3(1-\gamma^2)\xi_R^4-(3-2\gamma^2)\xi_R^2+1/2}h_{m+2j}^{(2)}(k\xi_R)j_{m+2l}(k\xi_R)\\&-\frac{ik}{4(1-\nu_h)}\int_0^{\gamma}\frac{\xi\sqrt{(\gamma^2-\xi^2)}}{(\xi^2-1/2)^2+\xi^2\sqrt{(1-\xi^2)}\sqrt{(\gamma^2-\xi^2)}}h_{m+2j}^{(2)}(k\xi)j_{m+2l}(k\xi)\,d\xi\\&-\frac{ik}{4(1-\nu_h)}\int_{\gamma}^{1}\frac{(\xi^2-\gamma^2)\xi^3\sqrt{(1-\xi^2)}}{(\xi^2-1/2)^4+\xi^4(1-\xi^2)(1-\xi^2)(\xi^2-\gamma^2)}h_{m+2j}^{(2)}(k\xi)j_{m+2l}(k\xi)\,d\xi,\ j\leq l\end{aligned} \tag{4.11}$$

$h_m^{(2)}(\ )$ is the spherical Bessel function of the third kind[9], and γ a material constant.

$$\gamma^2=h^2/k^2=\frac{1}{2}\frac{1-2\nu_h}{1-\nu_h} \tag{4.12}$$

ξ_R is the relative wave number of Rayleigh waves with respect to k, i. e. the positive root of

$$(\xi^2 - 1/2)^2 - \xi^2\sqrt{(\xi^2 - 1)}\sqrt{(\xi^2 - \gamma^2)} = 0 \tag{4.13}$$

For $j > l$ it is used that $A^m_{jl} = A^m_{lj}$. When several values of m are considered the computations are further reduced by the recurrence relation

$$A^m_{jl} = A^{m-2}_{j+1,\,l+1}. \tag{4.14}$$

If in addition to the contact stresses the half space surface is deformed by the wave $\mathring{w}(\eta, \theta)$, trigonometric and polynomial expansion similar to (2.10) and (4.4) give the expansion coefficients $\mathring{W}^m_l$, which must be added to the right side of (4.9) in order to obtain the total surface deformation.

In order to obtain a relation similar to (4.9) for the plate, the plate influence function $w^m_p(\eta, \xi)$ is expanded in terms of Legendre functions.

$$\frac{\mu_p}{1-\nu_p} w^m_p(\eta, \xi) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} B^m_{jl}(2m+4j+1)\frac{(2j)!}{(2m+2j)!}\xi P^m_{m+2j}(\sqrt{(1-\xi^2)})P^m_{m+2j}(0)$$

$$\times(2m+4l+1)\frac{(2l)!}{(2m+2l)!}P^m_{m+2l}(\sqrt{(1-\eta^2)})P^m_{m+2l}(0), \quad 0 \le \eta, \xi \le 1. \tag{4.15}$$

Although $w^m_p(\eta, \xi)$ is continuous, it follows from (3.27) that its first derivative exhibits a finite jump at $\eta = \xi$. In order to evaluate the coefficients B^m_{jl} accurately without excessive numerical effort the discontinuity was extracted and expanded analytically, while the remainder was expanded by use of the orthogonality relation (4.8) and Gaussian quadrature.

When the contact stress in the form (4.3) is substituted in (1.1) the resulting plate deformation has the expansion coefficients

$$W^m_l = -\frac{1-\nu_p}{\mu_p}\frac{\mu_h}{1-\nu_h}(2m+4l+1)\sum_{j=0}^{\infty} S^m_j B^m_{jl}. \tag{4.16}$$

The non-singular external load is accounted for through the expansion

$$q(\xi, \theta) = \sum_{m=0}^{\infty} q^m(\xi)\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{4.17}$$

where

$$\frac{1-\nu_h}{\mu_h} q^m(\xi) = \xi^m \sum_{k=0}^{\infty} N^m_k (1-\xi^2)^k. \tag{4.18}$$

The displacement contribution from the external load is found by use of the formula—[7], 3.12 (25),

$$\int_0^1 \xi^{m+1}(1-\xi^2)^k P^m_{m+2j}(\sqrt{(1-\xi^2)})\,d\xi = (-1)^m \frac{1}{2^{m+1}}\frac{(2m+2j)!}{(2j)!}\frac{k!}{(m+j+k+1)!}(k+3/2-j)_j \tag{4.19}$$

where $(z)_j = z(z+1)\dots(z+j-1)$ is Pochhammer's symbol. By combination of (4.15), (4.18) and (4.19) the expansion coefficients for plate displacements due to external load can be written as

$$W^m_l = \frac{1-\nu_p}{\mu_p}\frac{\mu_h}{1-\nu_h}(2m+4l+1)\sum_{j=0}^{\infty} B^m_{jl}\sum_{k=0}^{\infty} M^m_{jk} N^m_k \tag{4.20}$$

where

$$M_{jk}^m = \frac{(-1)^m}{2^{m+1}}(2m+4j+1)\frac{k!}{(m+j+k+1)!}(k+3/2-j)_j P_{m+2j}^m(0). \tag{4.21}$$

Now all contributions to the normal surface deformation of the half space and the plate have been expressed in terms of expansion coefficients. The contact condition eliminates the normal displacement and yields the following infinite systems of equations

$$(2m+4l+1)\sum_{j=0}^{\infty}\left[(-1)^{j+l}A_{jl}^m + \frac{1-\nu_p}{1-\nu_h}\frac{\mu_h}{\mu_p}B_{jl}^m\right]S_j^m$$

$$= -\overset{*}{W}_l^m + (2m+4l+1)\sum_{j=0}^{\infty}\left[\frac{1-\nu_p}{1-\nu_h}\frac{\mu_h}{\mu_p}B_{jl}^m\sum_{k=0}^{\infty}M_{jk}^m N_k^m\right], \quad m,l=0,1,2,\ldots \tag{4.22}$$

An approximate solution to (4.22) is obtained by suitable truncation of the infinite series. Examples are given in the next section.

In the limiting case of a rigid disc, $\mu_p \to \infty$, the only nonvanishing displacement expansion coefficients in (4.4) are W_0^0 and W_0^1. The relation between these coefficients and the load on the plate is determined by a special form of the principle of virtual work.

$$\int_0^a\int_0^{2\pi}[p(r,\theta) - \omega^2\rho_p H w(r,\theta)]w(r,\theta)r\,dr\,d\theta = 0. \tag{4.23}$$

The result amounts to the following replacement of the matrix B_{jl}^m.

$$(2m+4l+1)\frac{1-\nu_p}{1-\nu_h}\frac{\mu_h}{\mu_p}B_{jl}^m \to \begin{cases} -2\dfrac{\rho_h}{\rho_p}\dfrac{1}{k^2\tau(1-\nu_h)}, & j=l=0, m=0 \\ -\dfrac{16}{3}\dfrac{\rho_h}{\rho_p}\dfrac{1}{k^2\tau(1-\nu_h)}, & j=l=0, m=1 \\ 0, & \text{else.} \end{cases} \tag{4.24}$$

5. EXAMPLES

Axisymmetric vibrations of elastic and rigid plates have been treated in[2]. Here similar results for the rocking mode, $m = 1$, will be presented. The simplest case is that of a rigid disc investigated experimentally by Arnold *et al.*[10]. The angular amplitude is $|w^1(a)|/a = 1/2|W_0^1|$, and Fig. 2 gives the dimensionless parameter $|w^1(a)\mu_h a^2/M_y|$, where M_y is the moment amplitude, as a function of the dimensionless frequency $k = \omega a\sqrt{(\rho_h/\mu_h)}$. The parameter $b' = I_0/(\rho_h a^5)$, where I_0 is the moment of inertia about a diameter, characterizes the influence of the mass of the plate. In this particular case the inertial force is in phase with the displacement, and the influence of the mass can be accounted for explicitly. The agreement between theory and experiment can be improved somewhat by assuming a small but finite value of ν_p, e.g. $\nu_p = 0.05$, instead of the value $\nu_p = 0.0$ given in[10] and used in Fig. 2.

For low values of the frequency and rigid plates the results of the present theory do not deviate much from those of Bycroft[11] based on the assumption that the distribution of the contact stresses is the same as in the static case. When the frequency is increased or plates with finite stiffness are considered the stress distribution is radically changed and the full analysis is necessary. The effect of this is illustrated in Fig. 3 giving a dimensionless form of the average power input for a linear variation of the external load on three plates, $N_0^1 = 1$, i. e. $q(r,\theta) = r/a \cos\theta\, \mu_h/(1-\nu_h)$. The dimensionless form used is

$$\frac{1-\nu_h}{\mu_h}\sqrt{\left(\frac{\rho_h}{\mu_h}\right)}\frac{\langle P\rangle}{\pi a^2} = k\left\{\sum_{j=0}^{\infty}\frac{1}{4j+1}\mathrm{Re}\,[\overline{(iW_j^0)}S_j^0] + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2m+4j+1}\mathrm{Re}\,[\overline{(iW_j^m)}S_j^m)]\right\} \tag{5.1}$$

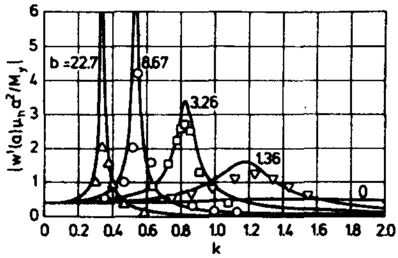

Fig. 2. Amplitude factor for rocking of rigid disc for various values of the inertial ratio $b' = I_0/(\rho_h a^5)$. Experimental results replotted from [10].

obtained by application of the following relation for the time average [3]

$$\langle \mathrm{Re}[F]\mathrm{Re}[f]\rangle = 1/2\,\mathrm{Re}[F\bar{f}]. \tag{5.2}$$

The parameter values are $\tau = 0.2$, $\nu_h = 0.25$, $\nu_p = 0.33$, $\rho_p/\rho_h = 3.0$ and the three stiffness ratios $\mu_p/\mu_h = 0.1, 10.0, \infty$.

It is seen that decreasing relative stiffness of the plate increases the power input at resonance and introduces an increasing number of secondary resonance phenomena. These findings are similar to those in [2] for axisymmetric vibrations of plates under uniform load. Figure 4 shows the normal displacements of the plates at $\theta = 0$ and phase angles 0, $\pi/3$ and $2\pi/3$ for the frequency $k = 2.5$. It is observed that the most flexible plate is close to having a node at the boundary at $\theta = 0$, and this appears to correspond to a minimum of the power input curve of Fig. 3.

Not only the behaviour of the plate but also the waves propagated through the half space are of interest, theoretically as well as in practical applications, e. g. to transducer calibration. Here attention will be restricted to the generation of surface waves. The surface displacement $w^m(\eta, 0)$ is given by the double integral (2.30), and when the contact stress has been expanded in (4.3) through the series of single integrals (4.7). For $\eta < 1$ this formula was used in connection with the orthogonal expansion (4.9) of the normal displacement to determine the matrix A_{ji}^m. For $\eta > 1$, i. e. on the half space surface outside the plate, use of a preconceived expansion for the displacements may obscure the basic features of the waves. Therefore the surface displacement $w^m(\eta, 0)$ for $\eta > 1$ is evaluated directly from (4.7). While the integrand in (4.10) is symmetric for large values of s, the integrand in (4.7) is antisymmetric. This necessitates a modification of the contour integration method used to reduce (4.10). In this case also a contribution from integration along the positive imaginary axis is present.

$$
\begin{aligned}
w^m(\eta, 0)/a = &-\frac{i\pi k}{4(1-\nu_h)}\frac{\xi_R^2(\xi_R^2-\gamma^2)\sqrt{(\xi_R^2-1)}}{3(1-\gamma^2)\xi_R^4-(3-2\gamma^2)\xi_R^2+1/2}H_m^{(2)}(\eta k\xi_R)\sum_{j=0}^{\infty}(-1)^j S_j^m j_{m+2j}(k\xi_R)\\
&-\frac{ik}{4(1-\nu_h)}\int_0^{\gamma}\frac{\xi\sqrt{(\gamma^2-\xi^2)}}{(\xi^2-1/2)^2+\xi^2\sqrt{(1-\xi^2)}\sqrt{(\gamma^2-\xi^2)}}H_m^{(2)}(\eta k\xi)\sum_{j=0}^{\infty}(-1)^j S_j^m j_{m+2j}(k\xi)\,\mathrm{d}\xi\\
&-\frac{ik}{4(1-\nu_h)}\int_{\gamma}^{1}\frac{(\xi^2-\gamma^2)\xi^3\sqrt{(1-\xi^2)}}{(\xi^2-1/2)^4+\xi^4(1-\xi^2)(\xi^2-\gamma^2)}H_m^{(2)}(\eta k\xi)\sum_{j=0}^{\infty}(-1)^j S_j^m j_{m+2j}(k\xi)\,\mathrm{d}\xi\\
&+\frac{k}{4(1-\nu_h)}i^{-m}\frac{2}{\pi}\int_0^{\infty}\frac{\xi\sqrt{(\xi^2+\gamma^2)}}{(\xi^2+1/2)^2-\xi^2\sqrt{(\xi^2+\gamma^2)}\sqrt{(\xi^2+1)}}K_m(\eta k\xi)\sum_{j=0}^{\infty}(-1)^j S_j^m j_{m+2j}(ik\xi)\,\mathrm{d}\xi,\\
&\eta > 1
\end{aligned}
\tag{5.3}
$$

$H_m^{(2)}(\;)$ is the Bessel function of the third kind [9]. The summations are truncated to the number of coefficients available from the solution of the contact problem, and the integrals are evaluated numerically.

Due to the oscillatory nature of the Bessel function $H_m^{(2)}(\eta k\xi)$ for large values of η the far

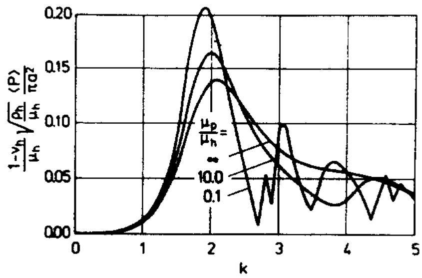

Fig. 3. Dimensionless power average for rigid and elastic plates with linear load distribution $q(r, \theta) = \mu_h/1 - \nu_h\ r/a \cos\theta$.

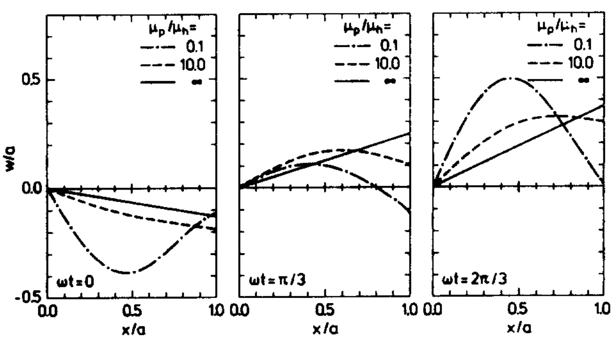

Fig. 4. Normal displacements at $\theta = 0$ for rigid and elastic plates at phase angles $\omega t = 0$, $\pi/3$, $2\pi/3$ and dimensionless frequency $k = 2.5$. Linear load distribution $q(r, \theta) = \mu_h/1 - \nu_h\ r/a \cos\theta$.

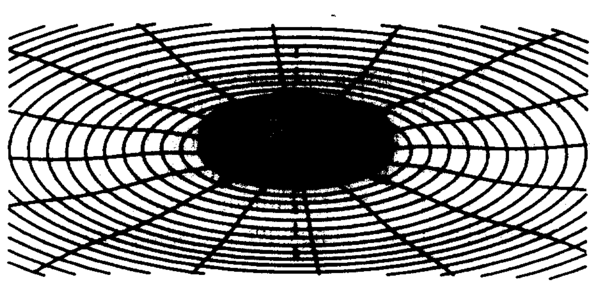

Fig. 5. Normal displacement field for $\mu_p/\mu_h = 10.0$, $k = 2.5$ and uniform load distribution, $q(r, \theta) = 1/2\, \mu_h/(1 - \nu_h)$.

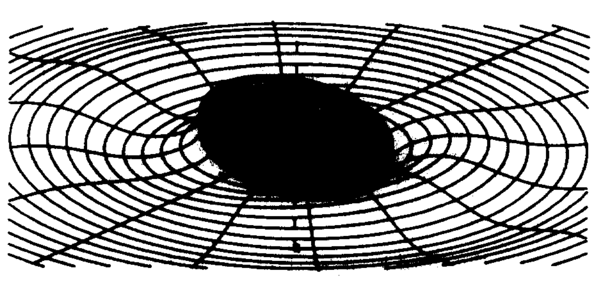

Fig. 6. Normal displacement field for $\mu_p/\mu_h = 10.0$, $k = 2.5$ and linear load distribution, $q(r, \theta) = \mu_h/(1 - \nu_h)\, r/a \cos\theta$.

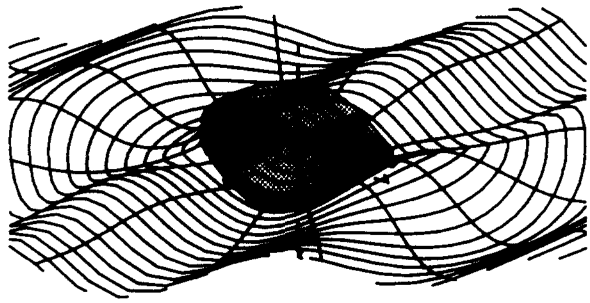

Fig. 7. Normal displacement field for $\mu_p/\mu_h = 10.0$, $k = 2.5$ and a plane Rayleigh wave, $\mathring{w}(r, \theta) = a/5 \exp(-ik_R r/a \cos\theta)$.

field will be dominated by the first, explicit term, which is a Rayleigh wave travelling away from the plate. The intensity of this wave is determined by a weighted sum of the contact stress expansion coefficients S_j^m, and as observed by Sezawa[4] the wave number is independent of the order m and equal to that of a plane Rayleigh wave. In the limit $k \to 0$ the static solution is obtained from the last integral. Computational results are shown in Figs. 5–7. In all three cases the parameters are those used above with the stiffness ratio $\mu_p/\mu_h = 10.0$. The frequency corresponds to $k = 2.5$.

Figure 5 shows axisymmetric vibration for uniformly distributed load, Fig. 6 shows rocking vibration for linear load distribution, and finally Fig. 7 shows the passage of a Rayleigh wave. In all three cases the influence of the integrals in (5.3) is restricted to a narrow zone around the plate.

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